CHAPTER 50
DEFLECTION

Joseph E. Shigley
Professor Emeritus
The University of Michigan
Ann Arbor, Michigan

Charles R. Mischke, Ph.D., P.E.
Professor Emeritus of Mechanical Engineering
Iowa State University
Ames, Iowa

GLOSSARY OF SYMBOLS

a	Dimension
A	Area
b	Dimension
C	Constant
D, d	Diameter
E	Young's modulus
F	Force
G	Shear modulus
I	Second moment of area
J	Second polar moment of area
k	Spring rate
K	Constant
ℓ	Length
M	Moment
$M(I)$	Moment relation, $(M/EI)_i$
N	Number

q	Unit load
Q	Fictitious force
R	Support reaction
T	Torque
U	Strain energy
V	Shear force
w	Unit weight
W	Total weight
x	Coordinate
y	Coordinate
δ	Deflection
θ	Slope, torsional deflection
ϕ	An integral
ψ	An integral

50.1 STIFFNESS OR SPRING RATE

The *spring rate* (also called *stiffness* or *scale*) of a body or ensemble of bodies is defined as the partial derivative of force (torque) with respect to colinear displacement (rotation). For a helical tension or compression spring,

$$F = \frac{d^4 G y}{8 D^3 N} \qquad \text{thus} \qquad k = \frac{\partial F}{\partial y} = \frac{d^4 G}{8 D^3 N} \tag{50.1}$$

where D = mean coil diameter
d = wire diameter
N = number of active turns

In a round bar subject to torsion,

$$T = \frac{GJ\theta}{\ell} \qquad \text{thus} \qquad k = \frac{\partial T}{\partial \theta} = \frac{GJ}{\ell} \tag{50.2}$$

and the tensile force in an elongating bar of any cross section is

$$F = \frac{AE\delta}{\ell} \qquad \text{thus} \qquad k = \frac{\partial F}{\partial \delta} = \frac{AE}{\ell} \tag{50.3}$$

If k is constant, as in these cases, then displacement is said to be linear with respect to force (torque). For contacting bodies with all four radii of curvature finite, the approach of the bodies is proportional to load to the two-thirds power, making the spring rate proportional to load to the one-third power. In hydrodynamic film bearings, the partial derivative would be evaluated numerically by dividing a small change in load by the displacement in the direction of the load.

50.2 DEFLECTION DUE TO BENDING

The relations involved in the bending of beams are well known and are given here for reference purposes as follows:

$$\frac{q}{EI} = \frac{d^4y}{dx^4} \tag{50.4}$$

$$\frac{V}{EI} = \frac{d^3y}{dx^3} \tag{50.5}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \tag{50.6}$$

$$\theta = \frac{dy}{dx} \tag{50.7}$$

$$y = f(x) \tag{50.8}$$

These relations are illustrated by the beam of Fig. 50.1. Note that the x axis is *positive* to the right and the y axis is *positive* upward. All quantities—loading, shear force, support reactions, moment, slope, and deflection—have the same sense as y; they are positive if upward, negative if downward.

50.3 PROPERTIES OF BEAMS

Table 50.1 lists a number of useful properties of beams having a variety of loadings. These must all have the same cross section throughout the length, and a linear relation must exist between the force and the deflection. Beams having other loadings can be solved using two or more sets of these relations and the principle of superposition.

In using Table 50.1, remember that the deflection at the center of a beam with off-center loads is usually within 2.5 percent of the maximum value.

50.4 COMPUTER ANALYSIS

In this section we will develop a computer method using numerical analysis to determine the slope and deflection of any simply supported beam having a variety of concentrated loads, including point couples, with any number of step changes in cross section. The method is particularly applicable to stepped shafts where the transverse bending deflections and neutral-axis slopes are desired at specified points.

The method uses numerical analysis to integrate Eq. (50.6) twice in a marching method. The first integration uses the trapezoidal rule; the second uses Simpson's rule (see Sec. 4.6). The procedure gives exact results.

Let us define the two successive integrals as

$$\phi = \int_0^x \frac{M}{EI}\, dx \qquad \psi = \int_0^x \phi \, dx \tag{50.9}$$

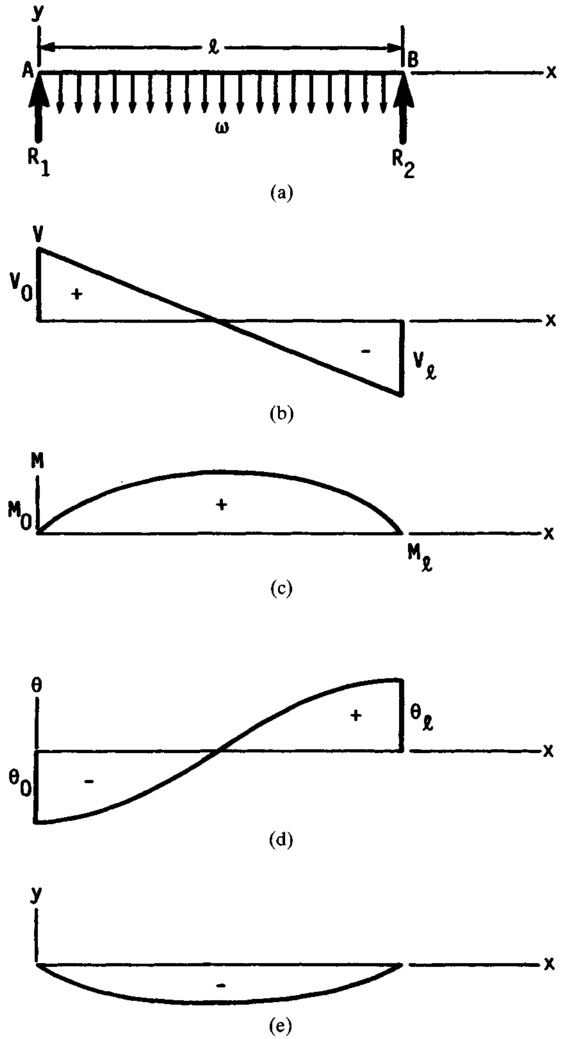


FIGURE 50.1 (*a*) Loading diagram showing beam supported at A and B with uniform load w having units of force per unit length, $R_1 = R_2 = w\ell/2$; (*b*) shear-force diagram showing end conditions; (*c*) moment diagram; (*d*) slope diagram; (*e*) deflection diagram.

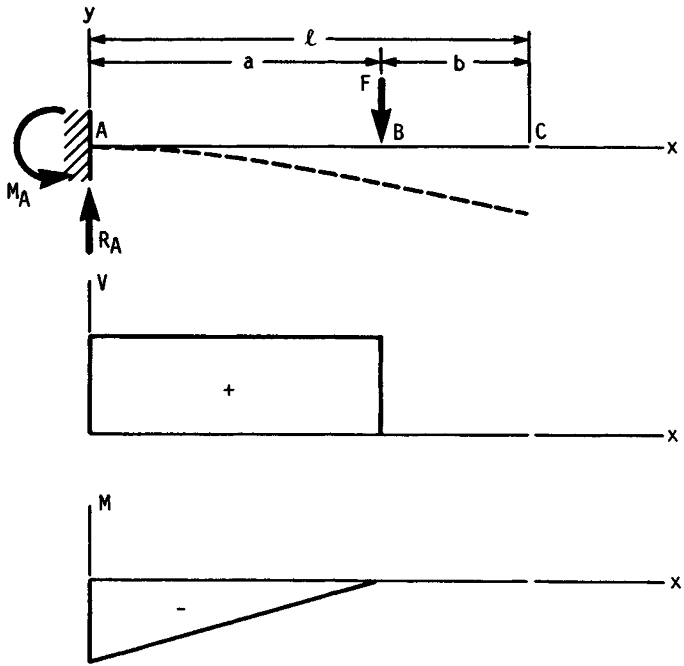
But from Eq. (50.7), the slope is

$$\theta = \frac{dy}{dx} = \int_0^x \frac{M}{EI}\,dx + C_1$$

$$= \phi + C_1 \qquad (a)$$

TABLE 50.1 Properties of Beams

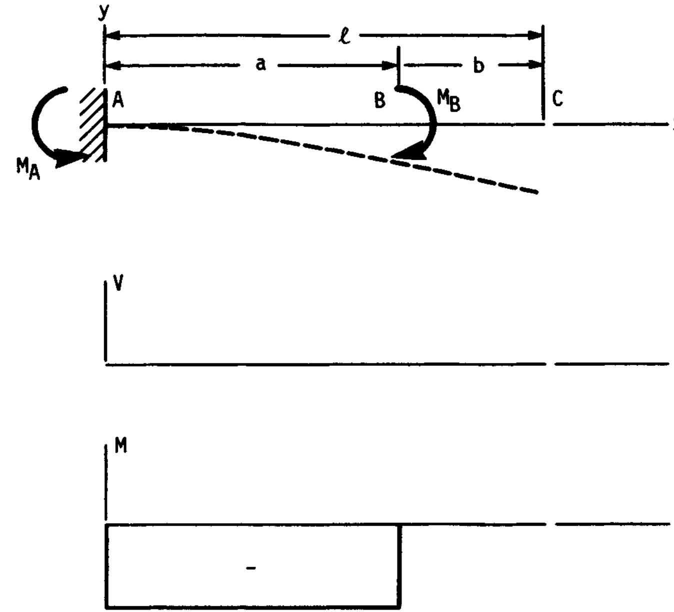
1. Cantilever—intermediate load



$R_A = F \qquad M_A = -Fa$

$$y_B = -\frac{Fa^3}{3EI} \qquad y_C = -\frac{Fa^3}{3EI}\left(1 + \frac{3b}{2a}\right)$$

2. Cantilever—intermediate couple

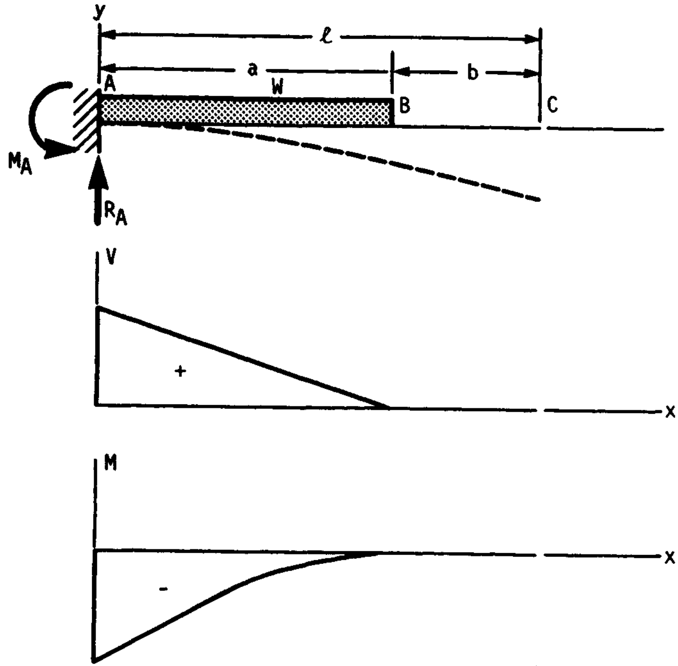


$V = 0 \qquad M_A = M$

$$y_B = -\frac{Ma^2}{2EI} \qquad y_C = -\frac{Ma^2}{2EI}\left(1 + \frac{2b}{a}\right)$$

TABLE 50.1 Properties of Beams (*Continued*)

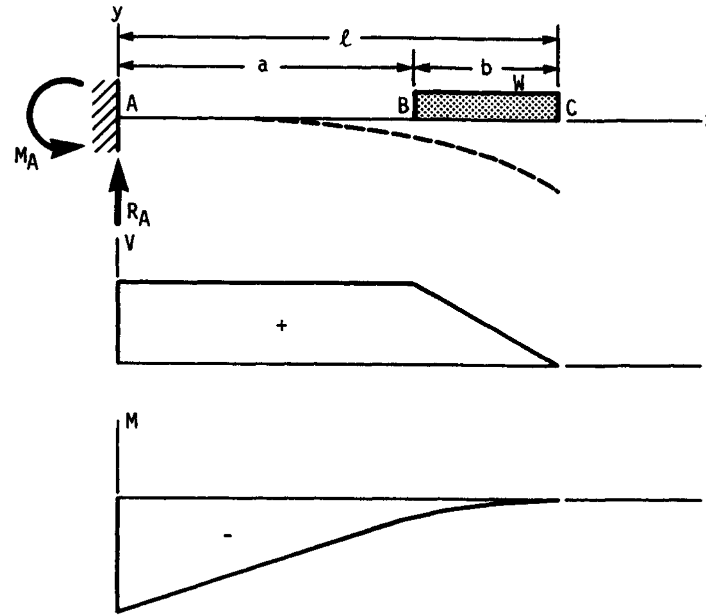
3. Cantilever—distributed load



$$R_A = W \qquad M_A = -\frac{Wa}{2}$$

$$y_B = -\frac{Wa^3}{8EI} \qquad y_C = -\frac{Wa^3}{8EI}\left(1 + \frac{4b}{3a}\right)$$

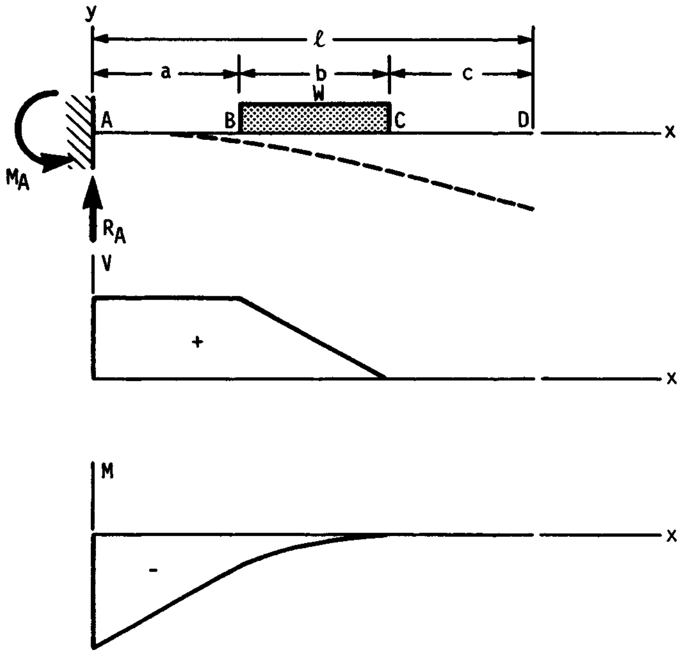
4. Cantilever—partial distributed load



$$R_A = W \qquad M_A = -W\left(a + \frac{b}{2}\right)$$

$$y_C = -\frac{W}{24EI}(8a^3 + 18a^2b + 12ab^2 + 3b^3)$$

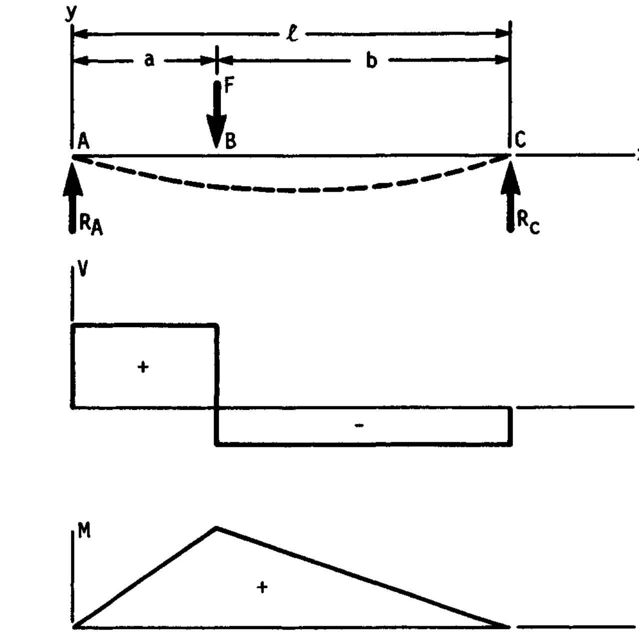
5. Cantilever—partial distributed load



$R_A = W \qquad M_A = -W\left(a + \frac{b}{2}\right)$

$$y_D = -\frac{W}{24EI}(8a^3 + 18a^2b + 12ab^2 + 3b^3 + 12a^2c + 12abc + 4b^2c)$$

6. Simple support—intermediate load

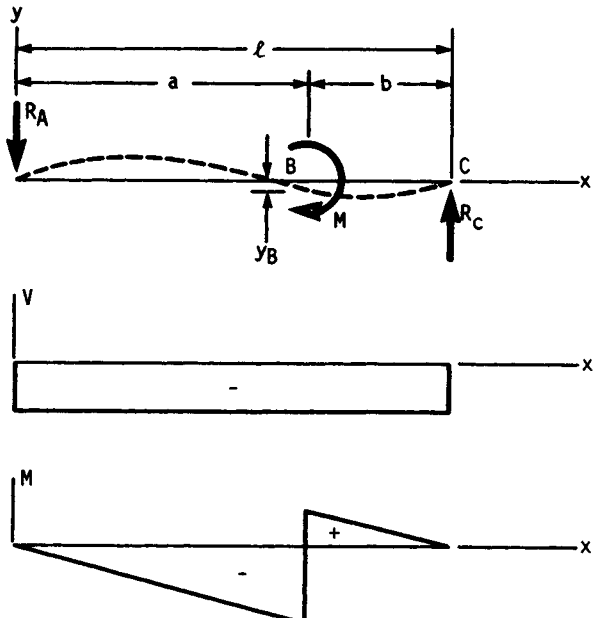


$R_A = \frac{Fb}{\ell} \qquad R_B = \frac{Fa}{\ell} \qquad M_B = \frac{Fab}{\ell}$

At center $y = -\frac{F\ell^3}{48EI}\left(\frac{3a}{\ell} - \left(\frac{4a}{\ell}\right)^3\right]$

TABLE 50.1 Properties of Beams (*Continued*)

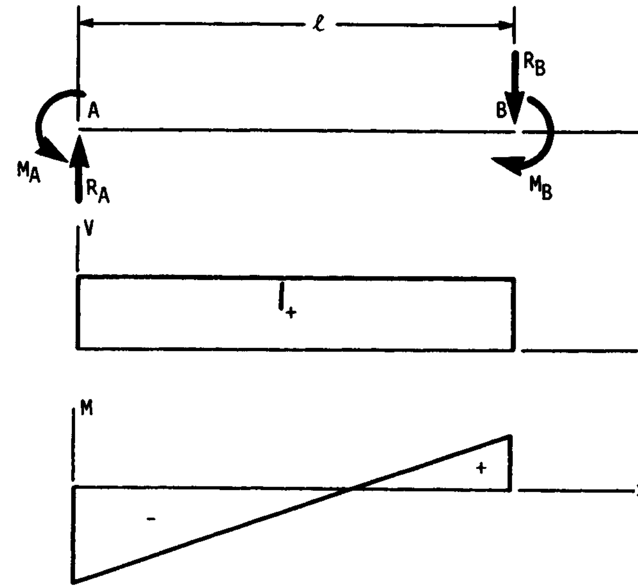
7. Simple support—intermediate couple



$$R_A = -R_C = -\frac{M}{\ell} \qquad M_{AB} = R_A x$$

$$y_B = -\frac{Mab}{3EI\ell}(a-b) \qquad a > b$$

8. Simple support—end moments

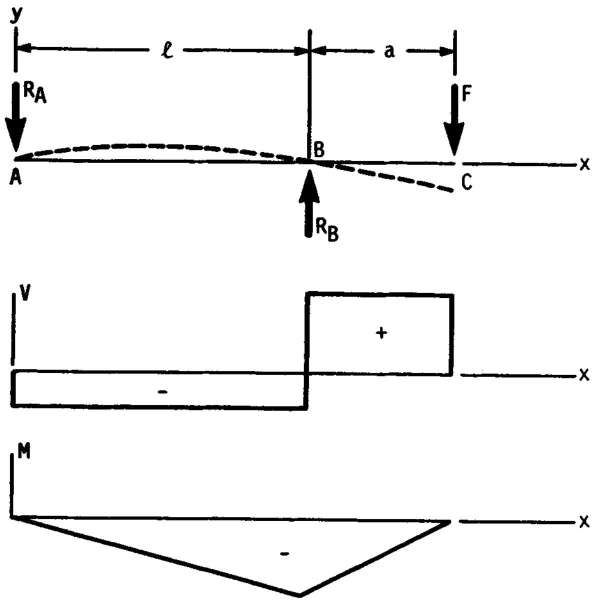


M_B positive, M_A negative, $|M_A| > |M_B|$

$$R_A = -R_B = -\frac{M_A + M_B}{\ell}$$

when $|M_A| = |M_B|$ $\qquad y_{max} = \frac{M\ell^2}{8EI}$

9. Simple support—overhung load

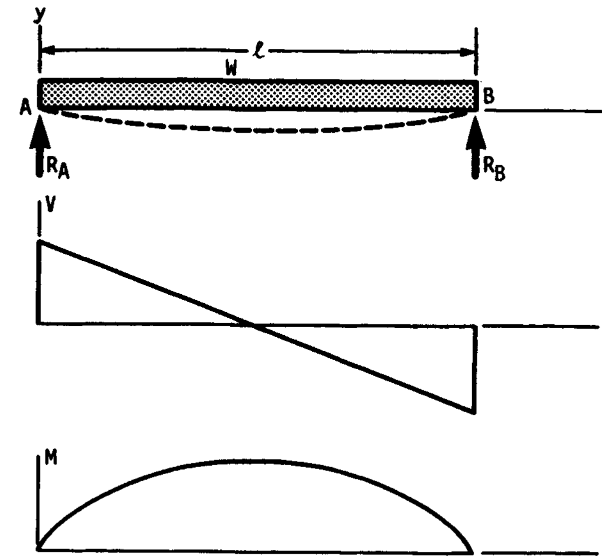


$$R_A = -\frac{Fa}{\ell} \qquad R_B = \frac{F}{\ell}(\ell + a)$$

$$M_B = -Fa$$

$$y_C = -\frac{Fa^2}{3EI}(\ell + a)$$

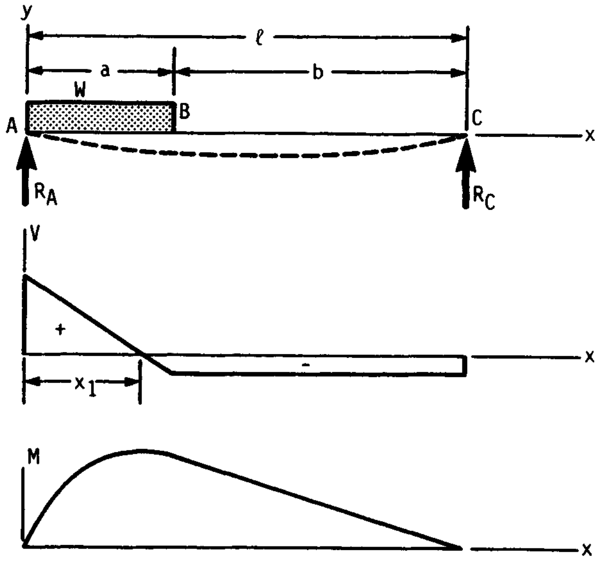
10. Simple support— uniform loading



$$R_A = R_B = \frac{W}{2} \qquad M_{max} = \frac{W\ell}{8}$$

$$y_{max} = -\frac{5W\ell^3}{384EI}$$

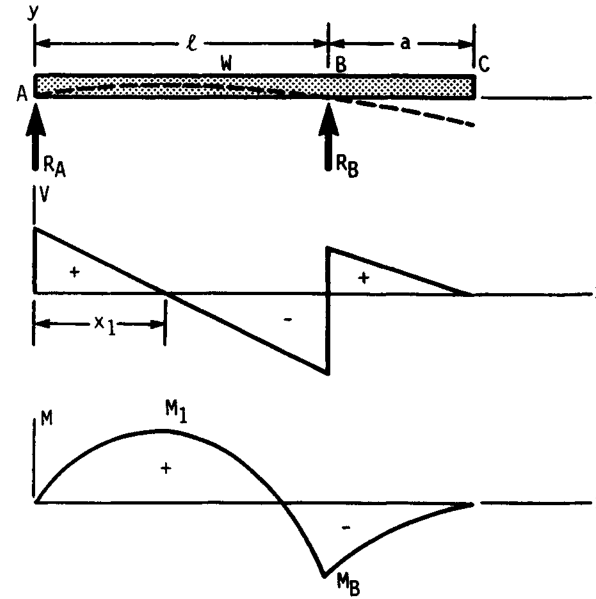
11. Simple support—partial uniform loading



$$R_A = \frac{W}{2\ell}(2\ell - a) \qquad R_B = \frac{Wa}{2\ell} \qquad x_1 = \frac{a}{2\ell}(2\ell - a)$$

At center $y = -\dfrac{Wa}{48EI}(a^2 + 2\ell^2) \qquad a < \dfrac{\ell}{2}$

12. Simple support—uniform loading, overhung

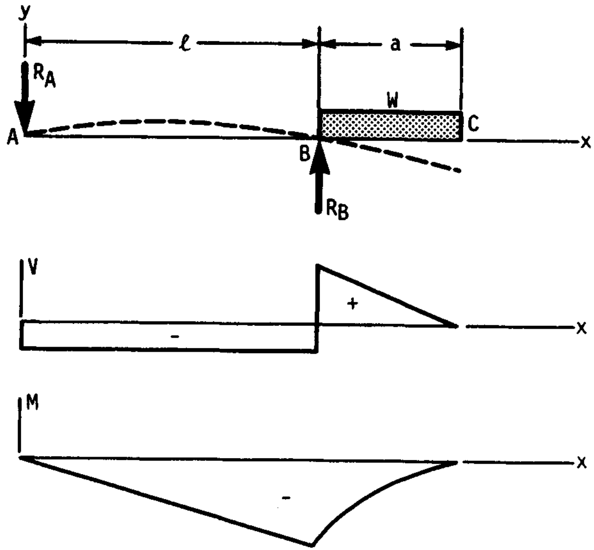


$$R_A = \frac{W}{2\ell}(\ell - a) \qquad R_B = \frac{W}{2\ell}(\ell + a) \qquad x_1 = \frac{1}{2\ell}(\ell^2 - a^2)$$

$$M_1 = \frac{W}{8\ell^2}(\ell + a)(\ell - a)^2 \qquad M_B = -\frac{Wa^2}{2(\ell + a)}$$

$$y_C = \frac{Wa}{24EI}(3a^2 + a\ell - \ell^2)$$

13. Simple support—overhung uniform load

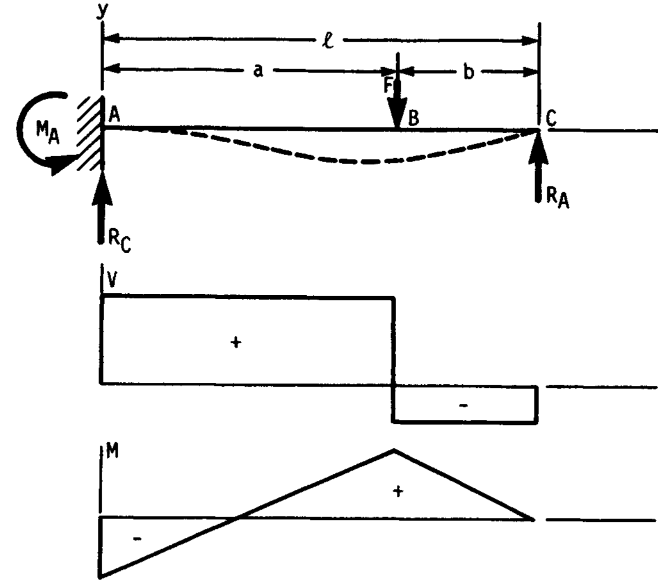


$$R_A = -\frac{Wa}{2\ell} \qquad R_B = \frac{W}{2\ell}(2\ell + a) \qquad M_{\max} = -\frac{Wa}{2}$$

$$y_{\max} = \frac{0.032\,Wa\ell^2}{EI} \quad \text{between supports}$$

$$y_C = -\frac{Wa^2}{24EI}(4\ell + 3a)$$

14. Fixed and simple support—intermediate load

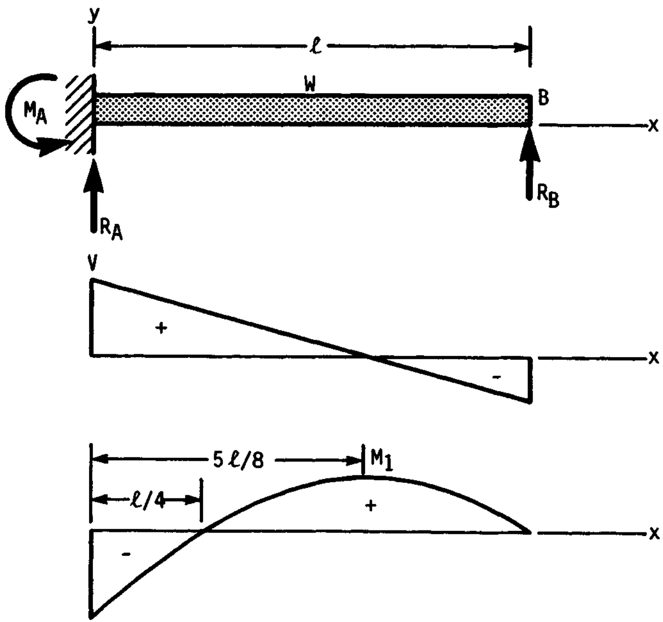


$$R_A = \frac{Fb}{2\ell^3}(3\ell^2 - b^2) \qquad R_C = \frac{Fa^2}{2\ell^3}(3\ell - a)$$

$$M_A = \frac{Fb}{2\ell^2}(b^2 - \ell^2) \qquad M_B = \frac{Fa^2b}{2\ell^3}(3\ell - a)$$

$$y_B = \frac{Fba^2}{12EI\ell^3}(3b^2\ell - 3\ell^3 + 3a\ell^2 - ab^2)$$

15. Fixed and simple support—uniform load

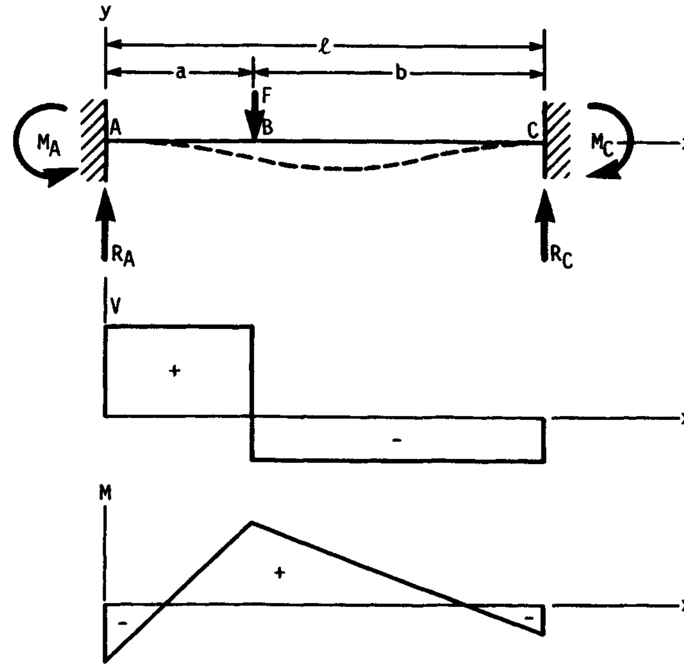


$$R_A = \frac{5W}{8} \qquad R_B = \frac{3W}{8}$$

$$M_A = -\frac{W\ell}{8} \qquad M_1 = \frac{9W\ell}{128}$$

$$y_{\max} = -\frac{W\ell^3}{185EI}$$

16. Fixed supports—intermediate load

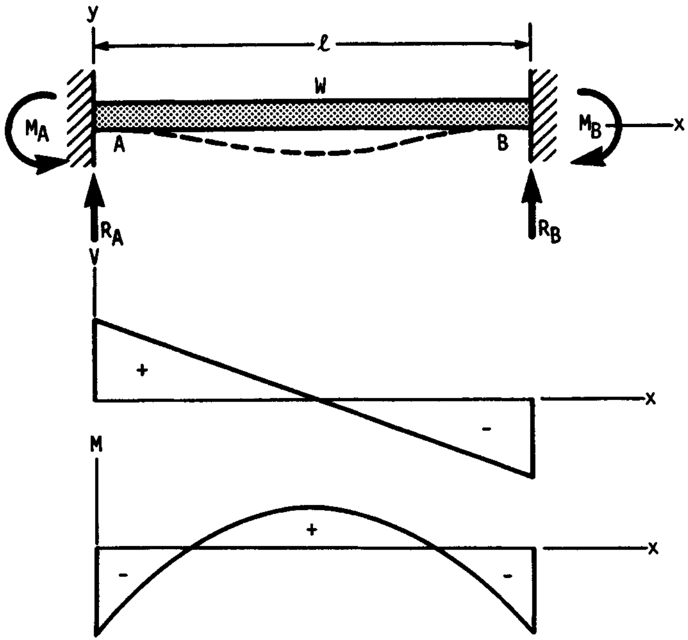


$$R_A = \frac{Fb^2}{\ell^3}(3a + b) \qquad R_B = \frac{Fa^2}{\ell^3}(3b + a)$$

$$M_A = -\frac{Fab^2}{\ell^2} \qquad M_B = \frac{Fab^2}{\ell^3}(3a + b - \ell)$$

$$M_C = -\frac{Fa^2b}{\ell^2} \qquad y_B = \frac{Fa^3b^2}{6EI\ell^3}(3a + b - 3\ell)$$

17. Fixed supports—uniform load



$$R_A = R_B = \frac{W}{2} \qquad M_A = M_B = -\frac{W\ell}{12}$$

$$\text{At } x = \frac{\ell}{2} \qquad M = \frac{W\ell}{24}$$

$$y_{\max} = -\frac{W\ell^3}{384EI}$$

A second integration gives

$$y = \psi + C_1 x + C_2 \tag{b}$$

It is convenient to write Eqs. (*a*) and (*b*) as

$$\theta = K(\phi + C_1) \tag{50.10}$$

$$y = K(\psi + C_1 x + C_2) \tag{50.11}$$

where K depends on the units used.

Locating supports at $x = a$ and $x = b$ and specifying zero deflection at these supports provides the two conditions for finding C_1 and C_2. The results are

$$C_1 = \frac{\psi_b - \psi_a}{x_a - x_b} \tag{50.12}$$

$$C_2 = \frac{x_b \psi_a - x_a \psi_b}{x_a - x_b} \tag{50.13}$$

Now we write the first of Eqs. (50.9) using the trapezoidal rule:

$$\phi_{i+2} = \phi_i + \frac{1}{2}\left[\left(\frac{M}{EI}\right)_{i+2} + \left(\frac{M}{EI}\right)_i\right](x_{i+2} - x_i) \tag{50.14}$$

Applying Simpson's rule to the second of Eqs. (50.9) yields

$$\psi_{i+4} = \psi_i + \frac{1}{6}(\phi_{i+4} + 4\phi_{i+2} + \phi_i)(x_{i+4} - x_i) \tag{50.15}$$

As indicated previously, these equations are used in a marching manner. Thus, using Eq. (50.14), we successively compute $\phi_1, \phi_3, \phi_5, \ldots$ beginning at x_1 and ending at x_N, where N is the number of M/EI values. Similarly, Eq. (50.15) is integrated successively to yield $\psi_1, \psi_5, \psi_9, \ldots, \psi_N$.

After these two integrations have been performed, the constants C_1 and C_2 can be found from Eqs. (50.12) and (50.13), and then Eqs. (50.10) and (50.11) can be solved for the deflection and slope. These terms will have the same indices as the integral ψ. See Chap. 37, pp. 37.5–37.8 for a shaft analysis example.

The details of the method are best explained by an example. The shaft of Fig. 50.2 has all points of interest designated by the station letters *A, B, C*,. . . . These points must include

- Location of all supports and concentrated loads
- Location of cross-sectional changes
- Location of points at which the deflection and slope are desired

Refer now to Table 50.2 and note that coordinates x tabulated in column 2 correspond to each station. Note also the presence of additional x coordinates; these are selected as halfway stations.

Column 4 of Table 50.2 shows that two M/EI values must be computed for each x coordinate. These are needed to account for the fact that M/EI has an abrupt change at every shoulder or change in cross section.

The indices $i = 1, 2, 3, \ldots, N$ in Eqs. (50.14) and (50.15) correspond to the M/EI values and are shown in column 3 of Table 50.2. A program in BASIC is shown in Fig. 50.3. Note that the term M(I) is used for $(M/EI)_i$.

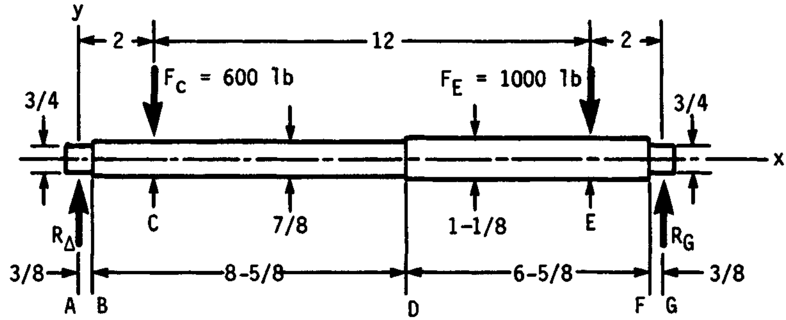


FIGURE 50.2 Simply supported stepped shaft loaded by forces F_C and F_E and supported by bearing reactions R_A and R_G. All dimensions in inches.

50.5 ANALYSIS OF FRAMES

Castigliano's theorem is introduced in Chap. 16, and the energy equations needed for its use are listed in Table 16.2. The method can be used to find the deflection at any point of a frame such as the one shown in Fig. 50.4. For example, the deflection δ_C at C in the direction of F_2 can be found using Eq. (16.2) as

$$\delta_C = \frac{\partial U}{\partial F_2} \tag{50.16}$$

where U = the strain energy stored in the entire frame due to all the forces. If the deflection is desired in another direction or at a point where no force is acting, then a fictitious force Q is added to the system at that point and in the direction in which the deflection is desired. After the partial derivatives have been found, Q is equated to zero, and the remaining terms give the wanted deflection.

The first step in using the method is to make a force analysis of each member of the frame. If Eq. (*a*) is to be solved, then the numerical values of F_1 and F_2 can be used in the force analysis, but the value of F_2 must *not* be substituted until after each member has been analyzed and the partial derivatives obtained. The following example demonstrates the technique.

Example 1. Find the downward deflection of point D of the frame shown in Fig. 50.5.

Solution. A force analysis of the system gives an upward reaction at E of $R_E = 225 + 3F_2$. The reaction at A is downward and is $R_A = 75 - 2F_2$.

The strain energy for member CE is

$$U_{CE} = \frac{R_A^2 \ell}{2AE} \tag{1}$$

The partial deflection is taken with respect to F_2 because the deflection at D in the direction of F_2 is desired. Thus

$$\frac{\partial U_{CE}}{\partial F_2} = \frac{2R_A \ell}{2AE} \frac{\partial R_A}{\partial F_2} \tag{2}$$

TABLE 50.2 Summary of Beam Computations†

Station (1)	x (2)	N (3)	M/EI (4)	ϕ (5)	ψ (6)	y (7)	θ (8)
A	0	0 1	0 0	0	0	0	−1.028E-02
	0.188	2 3	261.6 261.6	24.59			
B	0.375	4 5	523.2 282.4	98.0	12.27	−3.8444E-03	−1.019E-02
	1.188	6 7	894.6 894.6	576.4			
C	2	8 9	1 506 1 506	1 551	1 083	−0.0195	−8.733E-03
	5.5	10 11	1 708.7 1 708.7	7 177			
D	9	12 13	1 911.4 699.5	13 512	52 149	−4.0408E-02	3.228E-03
	11.5	14 15	752.5 752.5	15 327			
E	14	16 17	805.5 805.5	17 274	128 894	−1.5084E-02	6.990E-03
	14.813	18 19	478.1 478.1	17 796			
F	15.625	20 21	151.0 764.7	18 052	157 741	−2.9488E-03	7.768E-03
	15.813	22 23	382.3 382.3	18 159			
G	16	24 25	0 0	18 195	164 546	0	7.911E-03

†The units are in for x, lb·in for M, Mpsi for E, in^4 for I, in for y, and rad for θ.

```
10 PRINT "YOU MAY USE EITHER U.S. CUSTOMARY UNITS IN THIS PROGRAM"
20 PRINT "OR METRIC UNITS.  IF U.S. CUSTOMARY UNITS ARE USED"
30 PRINT "M IS IN INCH-POUNDS, E IN MPSI, AND I IN INCHES TO"
40 PRINT "THE FOURTH POWER.  IF METRIC UNITS ARE USED, M IS IN"
50 PRINT "NEWTON-METERS, E IN GPA, AND I IN CENTIMETERS TO THE"
60 PRINT "FOURTH POWER."
70 PRINT "WILL YOU USE METRIC UNITS (Y OR N)";U$
80 INPUT U$
90 IF U$= "Y" THEN 100 ELSE 110
100 K = .0001:GOTO 120
110 K = .000001
120 DIM M(65),X(65),PHI(65),PSI(65),Y(65),THETA(65)
130 INPUT "N=";N: FOR I = 1 TO N
140 INPUT "M=";M(I): LPRINT "M("I")="M(I)
150 NEXT I
160 FOR I = 1 TO N STEP 2
170 INPUT "X=";X(I): LPRINT "X("I")="X(I)
180 NEXT I
190 STOP
200 LPRINT "PHI( 1 )="PHI(1)
210 FOR I = 1 TO (N-2) STEP 2
220 PHI(I+2) = PHI(I) + ((M(I+1) + M(I))*(X(I+2) - X(I))*.5)
230 LPRINT "PHI("I+2")="PHI(I+2)
240 NEXT I
250 LPRINT "PSI( 1 )="PSI(1)
260 FOR I = 1 TO (N-4) STEP 4
270 PSI(I+4)=PSI(I)+(((PHI(I+4)+(4*PHI(I+2))+PHI(I))*(X(I+4)-X(I)))/6)
280 LPRINT "PSI("I+4")="PSI(I+4)
290 NEXT I
300 PRINT "SPECIFY VALUES OF X AND PSI AT SUPPORT A"
310 INPUT "X=";A    : LPRINT "X(A)="A
320 INPUT "PSI=";PSIA  : LPRINT "PSI(A)="PSIA
330 LPRINT
340 PRINT "SPECIFY VALUES OF X AND PSI AT SUPPORT B"
350 INPUT "X=";B    : LPRINT "X(B)="B
360 INPUT "PSI=";PSIB  : LPRINT "PSI(B)="PSIB
370 LPRINT
380 C1  = (PSIB - PSIA)/(A-B)
390 LPRINT "C(1)="C1
400 C2 =((B*PSIA)-(A*PSIB))/(A-B)
410 LPRINT "C(2)="C2
420 LPRINT
430 FOR I = 1 TO N STEP 4
440 IF X(I) = A THEN 450 ELSE 460
450 Y(I)=0: GOTO 490
460 IF X(I)=B THEN 470 ELSE 480
470 Y(I) = 0: GOTO 490
480 Y(I) =(PSI(I) + (C1  *X(I)) + C2)*K
490 LPRINT "Y("I")="Y(I)
500 NEXT I
510 LPRINT
520 FOR I = 1 TO N STEP 4
530 THETA(I) =(PHI(I) + C1)*K
540 LPRINT "THETA("I")="THETA(I)
550 NEXT I
560 END
```

FIGURE 50.3 Beam problem programmed in BASIC.

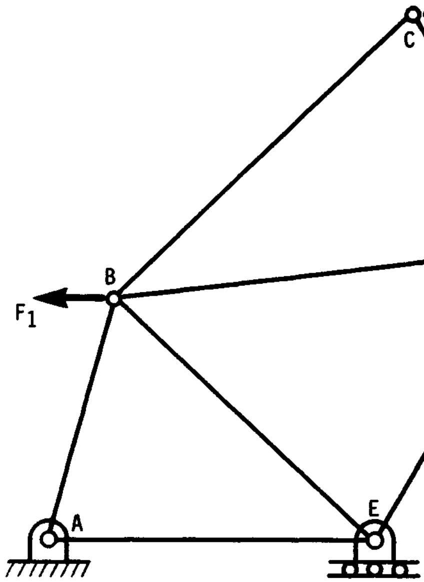


FIGURE 50.4 Frame loaded by three forces.

Also,

$$\frac{\partial R_A}{\partial F_2} = -2$$

Thus Eq. (2) becomes

$$\frac{\partial U_{CE}}{\partial F_2} = \frac{(75 - 2F_2)(30)}{0.2E}(-2) = \frac{37\,500}{E} \tag{3}$$

Note that we were able to substitute the value of F_2 in Eq. (3) because the partial derivative had been taken.

The strain energy stored in member *ABCD* will have to be computed in three parts because of the change in direction of the bending moment diagram at points *B* and *C*. For part *AB*, the moment is

$$M_{AB} = R_A x = (75 - 2F_2)x$$

The strain energy is

$$U_{AB} = \int_0^6 \frac{M_{AB}^2}{2EI}\,dx \tag{4}$$

Taking the partial derivative with respect to F_2 as before gives

$$\frac{\partial U_{AB}}{\partial F_2} = \int_0^6 \frac{2M_{AB}}{2EI}\,\frac{\partial M_{AB}}{\partial F_2}\,dx \tag{5}$$

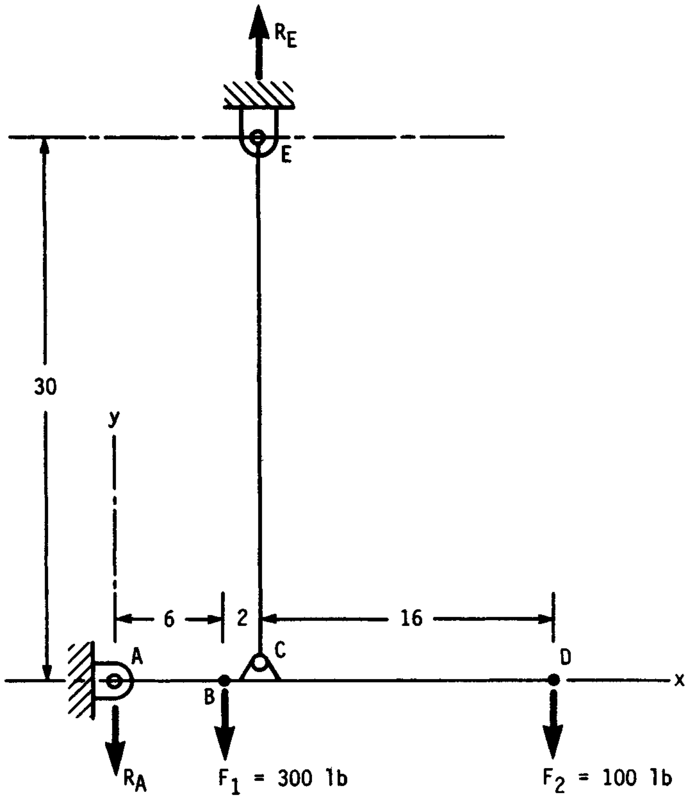


FIGURE 50.5 Frame loaded by two forces. Dimensions in inches: $A_{CE} = 0.20$ in^2; $I_{AD} = 0.18$ in^4; $E = 30 \times 10^6$ psi.

But

$$\frac{\partial M_{AB}}{\partial F_2} = -2x \tag{6}$$

Therefore, Eq. (5) may be written

$$\frac{\partial U_{AB}}{\partial F_2} = \frac{1}{EI}\int_0^6 x(75 - 2F_2)(-2x)\,dx$$

$$= \frac{1}{0.18E}\int_0^6 250x^2\,dx = \frac{100\,000}{E} \tag{7}$$

where the value of F_2 again has been substituted after taking the partial derivative.

For section BC, we have

$$M_{BC} = R_A x - F_1(x - 6) = 1800 - 225x - 2F_2 x$$

$$\frac{\partial M_{BC}}{\partial F_2} = -2x$$

$$\frac{\partial U_{BC}}{\partial F_2} = \int_6^8 \frac{2M_{BC}}{2EI}\frac{\partial M_{BC}}{\partial F_2}\,dx$$

$$= \frac{1}{EI}\int_6^8 (1800 - 225x - 2F_2x)(-2x)\,dx$$

$$= \frac{1}{0.18E}\int_6^8 (-3600x + 850x^2)\,dx = \frac{145\,926}{E}$$

Finally, section CD yields

$$M_{CD} = -(24 - x)F_2 \qquad \frac{\partial M_{CD}}{\partial F_2} = -(24 - x)$$

$$\frac{\partial U_{CD}}{\partial F_2} = \int_8^{24} \frac{2M_{CD}}{2EI}\frac{\partial M_{CD}}{\partial F_2}\,dx$$

$$= \frac{1}{EI}\int_8^{24} F_2(24 - x)^2\,dx$$

$$= \frac{1}{0.18E}\int_8^{24} (57\,600 - 4800x + 100x^2)\,dx$$

$$= \frac{758\,519}{E}$$

Then

$$y_D = \frac{\partial U_{CE}}{\partial F_2} + \frac{\partial U_{AB}}{\partial F_2} + \frac{\partial U_{BC}}{\partial F_2} + \frac{\partial U_{CD}}{\partial F_2}$$

$$= \frac{1}{30(10)^6}(37\,500 + 100\,000 + 145\,926 + 758\,519)$$

$$= 0.0347 \text{ in} \qquad \text{(when rounded)}$$

50.5.1 Redundant Members

A frame consisting of one or more redundant members is statically indeterminate because the use of statics is not sufficient to determine all the reactions. In this case, Castigliano's theorem can be used first to determine these reactions and second to determine the desired deflection.

Let R_1, R_2, and R_3 be a set of three indeterminate reactions. The deflection at the supports must be zero, and so Castigliano's theorem can be written three times. Thus

$$\frac{\partial U}{\partial R_1} = 0 \qquad \frac{\partial U}{\partial R_2} = 0 \qquad \frac{\partial U}{\partial R_3} = 0 \tag{50.17}$$

and so the number of equations to be solved is the same as the number of indeterminate reactions.

In setting up Eqs. (50.17), *do not* substitute the numerical value of the particular force corresponding to the desired deflection. This force symbol must appear in the reaction equations because the partial derivatives must be taken with respect to this force when the deflection is found. The method is illustrated by the following example.

Example 2. Find the downward deflection at point D of the frame shown in Fig. 50.6.

Solution. Choose R_B as the statically indeterminate reaction. A static force analysis then gives the remaining reactions as

$$R_A = R_C = 0.625(F - R_B) \tag{1}$$

The frame consists only of tension members, so the strain energy in each member is

$$U_{AD} = U_{DC} = \frac{R_A^2 \ell_{AD}}{2A_{AD}E} \qquad U_{BD} = \frac{R_B^2 \ell_{BD}}{2A_{BD}E} \tag{2}$$

Using Eq. (50.17), we now write

$$0 = \frac{\partial U}{\partial R_B} = \frac{2R_A \ell_{AD}}{A_{AD}E}\frac{\partial R_A}{\partial R_B} + \frac{R_B \ell_{BD}}{A_{BD}E}\frac{\partial R_B}{\partial R_B} \tag{3}$$

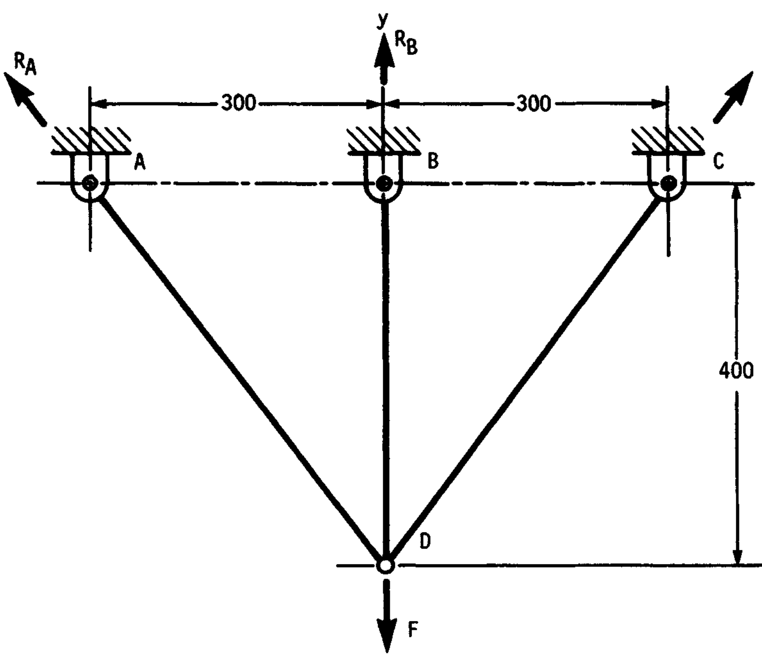


FIGURE 50.6 Frame loaded by a single force. Dimensions in millimeters: $A_{AD} = A_{CD} = 2\text{ cm}^2$, $A_{BD} = 1.2\text{ cm}^2$, $E = 207$ GPa, $F = 20$ kN.

Equation (1) gives $\partial R_A/\partial R_B = -0.625$. Also, $\partial R_B/\partial R_B = 1$. Substituting numerical values in Eq. (3), except for F, gives

$$\frac{2(0.625)(F - R_B)(500)(-0.625)}{2(207)} + \frac{R_B(400)(1)}{1.2(207)} = 0 \tag{4}$$

Solving gives $R_B = 0.369F$. Therefore, from Eq. (1), $R_A = R_C = 0.394F$. This completes the solution of the case of the redundant member. The next problem is to find the deflection at D.

Using Eq. (2), again we write

$$y_D = \frac{\partial U}{\partial F} = \frac{2R_A\ell_{AD}}{A_{AD}E}\frac{\partial R_A}{\partial F} + \frac{R_B\ell_{BD}}{A_{BD}E}\frac{\partial R_B}{\partial F} \tag{5}$$

For use in this equation, we note that $\partial R_A/\partial F = 0.394$ and $\partial R_B/\partial F = 0.369$. Having taken the derivatives, we can now substitute the numerical value of F. Thus Eq. (5) becomes†

$$\begin{aligned} y_D &= \left\{\frac{2[0.394(20)](500)(0.394)}{2(207)} + \frac{[0.369(20)](400)(0.369)}{1.2(207)}\right\}10^{-2} \\ &= 0.119 \text{ mm} \end{aligned}$$

If care is taken to refrain from substituting numerical values for reactions or forces until after partial derivatives are taken, Castigliano's theorem is applicable to statically indeterminate frames containing redundant members.

† In general, when using metric quantities, prefixed units are chosen so as to produce number strings of not more than four members. Thus some preferred units in SI are MPa (N/mm^2) for stress, GPa for modulus of elasticity, mm for length, and, say, cm^4 for second moment of area.

People are sometimes confused when they encounter an equation containing a number of mixed units. Suppose we wish to solve a deflection equation of the form

$$y = \frac{64F\ell^3}{3\pi d^4 E}$$

where $F = 1.30$ kN, $\ell = 300$ mm, $d = 2.5$ cm, and $E = 207$ GPa. Form the equation into two parts, the first containing the numbers and the second containing the prefixes. This converts everything to base units, including the result. Thus,

$$y = \frac{64(1.30)(300)^3}{3\pi(2.5)^4(207)}\,\frac{(\text{kilo})(\text{milli})^3}{(\text{centi})^4(\text{giga})}$$

Now compute the numerical value of the first part and substitute the prefix values in the second. This gives

$$\begin{aligned} y &= (29.48 \times 10^3)\left[\frac{10^3(10^{-3})^3}{(10^{-2})^4(10^9)}\right] = 29.48 \times 10^{-4} \text{ m} \\ &= 2.948 \text{ mm} \end{aligned}$$

Note that we multiplied the result by 10^3 mm/m to get the answer in millimeters. When this approach is used with Eq. (5), it is found that the result must be multiplied by $(10)^{-2}$ to get y in millimeters.